Separating complexity classes of LCL problems on grids

Katalin Berlow, Anton Bernshteyn, Clark Lyons, Felix Weilacher

SEALS 2025

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Schreier Graphs

Definition:

Let $\Gamma = \langle S \rangle \curvearrowright X$ be a Borel action of a finitely generated group on a standard Borel space. The **Schreier graph** of this action is the Borel graph $G \subseteq X \times X$ where we have $(x, y) \in G$ iff $\exists s \in S \; s \cdot x = y$.

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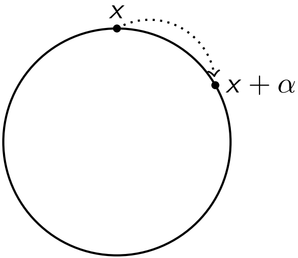

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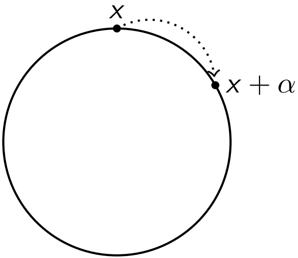

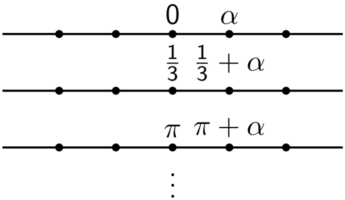

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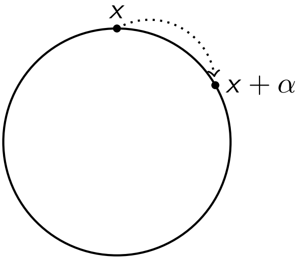

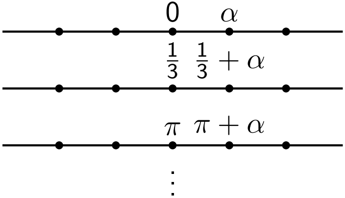

Each connected component will look like a copy of the Cayley graph of $\mathbb{Z}$ because the action is free.

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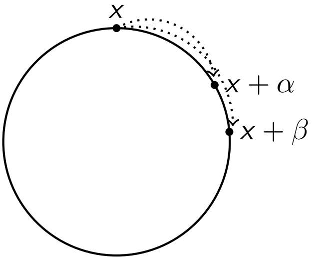

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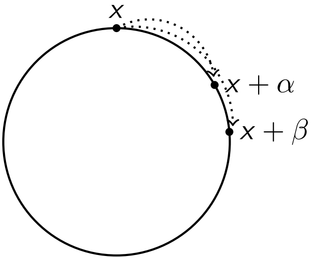

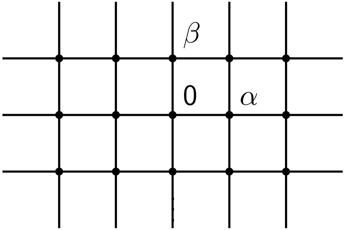

Proper coloring is an LCL

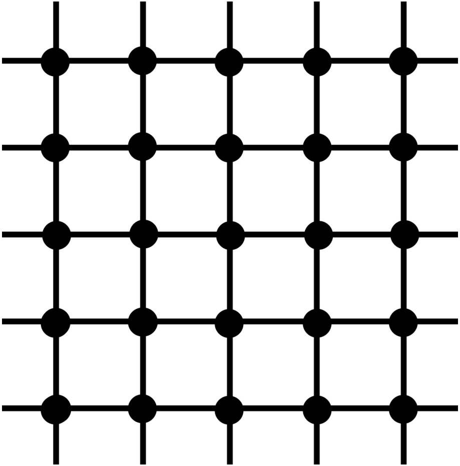

Figure: Proper coloring on $\mathbb{Z}^2$

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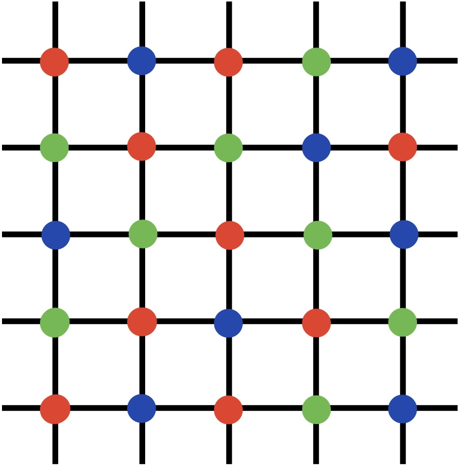

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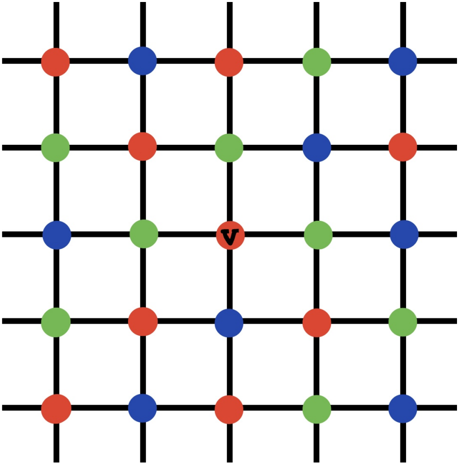

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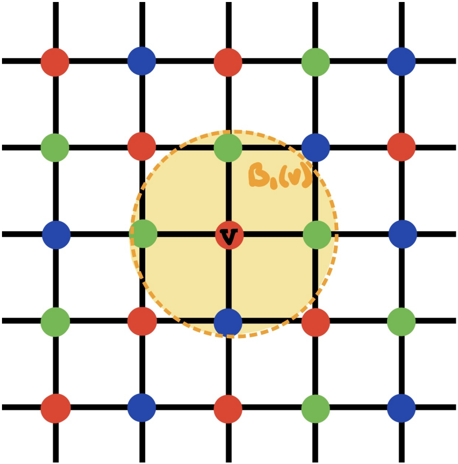

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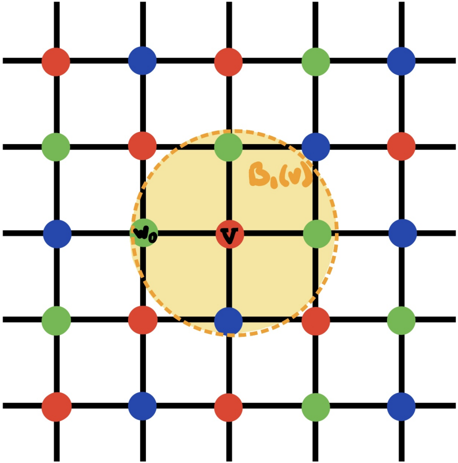

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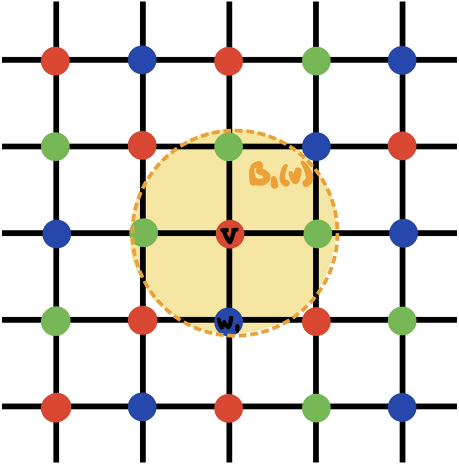

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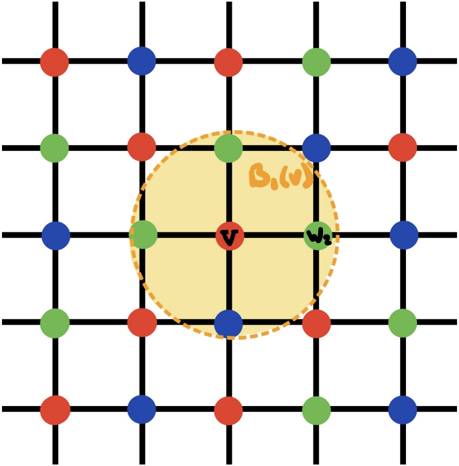

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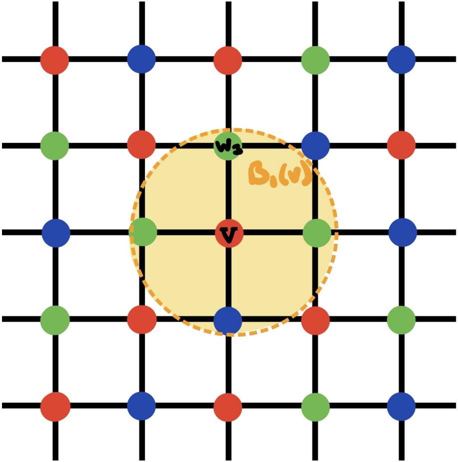

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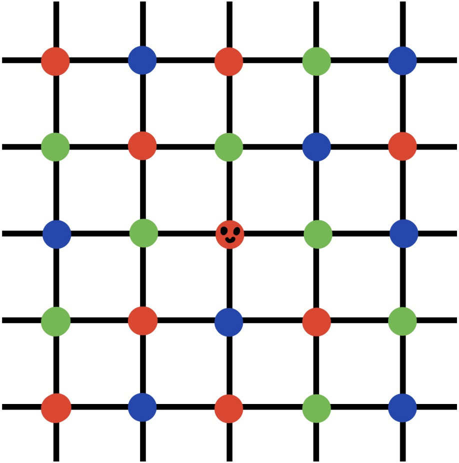

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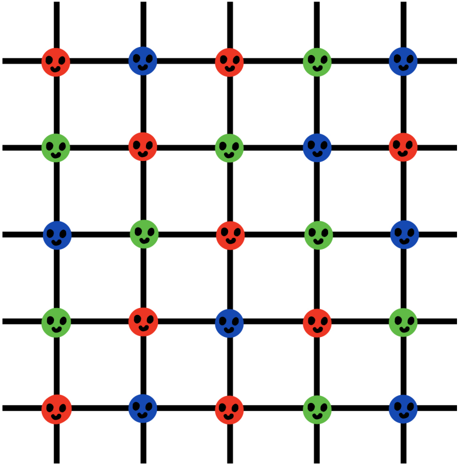

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Locally Checkable Labeling Problems

Idea: Label the vertices of G according to some *rule* which can be verified *locally*.

Examples of LCLs:

- proper vertex coloring
- proper edge coloring
- matchings
- sinkless orientation
- Wang tiling

Nonexamples of LCLs:

- Hamiltonian cycle
- spanning trees

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Different Notions of Definability

Definition (Descriptive complexity classes)

For an LCL Π, we say:

- $\Pi \in \mathrm{BOREL}(\Gamma)$ iff every free Borel action $\Gamma \curvearrowright (X, \mathcal{B})$ on a standard Borel space admits a **Borel** solution.
- $\Pi \in \mathrm{MEAS}(\Gamma)$ iff every free Borel action $\Gamma \curvearrowright (X, \mu)$ on a standard probability space admits a μ-**measurable** solution.
- $\Pi \in \mathrm{BAIRE}(\Gamma)$ iff every free Borel action $\Gamma \curvearrowright (X, \tau)$ on a Polish space admits a **Baire measurable** solution.

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Example: Proper $2n$-coloring is in $\mathrm{MEAS}(\mathbb{F}_n)$ and $\mathrm{BAIRE}(\mathbb{F}_n)$ by Conley–Marks–Tucker-Drob (2016) but not in $\mathrm{BOREL}(\mathbb{F}_n)$ by Marks (2013).

Complexity Classes

BOREL, MEAS, BAIRE, FIID, FFIID, ...

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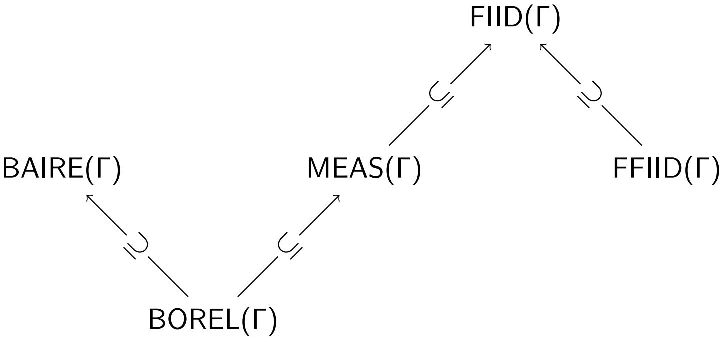

Figure: The trivial inclusions for any Γ.

Previous Results

Grebík–Rozhoň (2021) have shown:

$$\mathsf{BOREL}(\mathbb{Z}) = \mathsf{BAIRE}(\mathbb{Z}) = \mathsf{MEAS}(\mathbb{Z}) = \mathsf{FIID}(\mathbb{Z}) = \mathsf{FFIID}(\mathbb{Z})$$

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The following was left open:

- (Grebík–Rozhoň) Is $\mathsf{BOREL}(\mathbb{Z}^n) \subseteq \mathsf{MEAS}(\mathbb{Z}^n)$ strict for $n > 1$?
- Does $\mathsf{MEAS}(\Gamma) \subseteq \mathsf{BAIRE}(\Gamma)$ hold for all Γ?
- Is $\mathsf{FFIID}(\Gamma) = \mathsf{FIID}(\Gamma)$ for every Γ?

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Problems on $\mathbb{Z}^d$

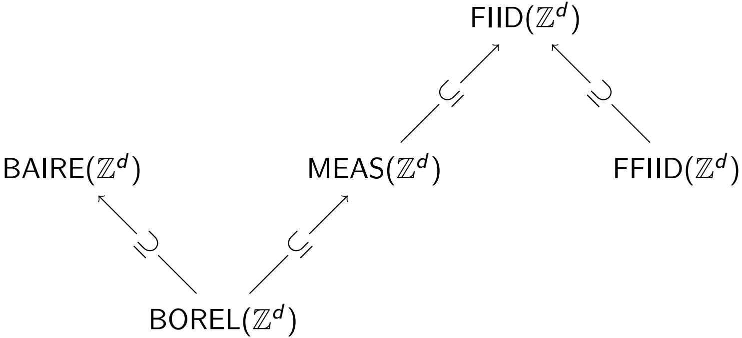

Figure: Complexity classes of LCLs on $\mathbb{Z}^d$, $d \geq 2$.

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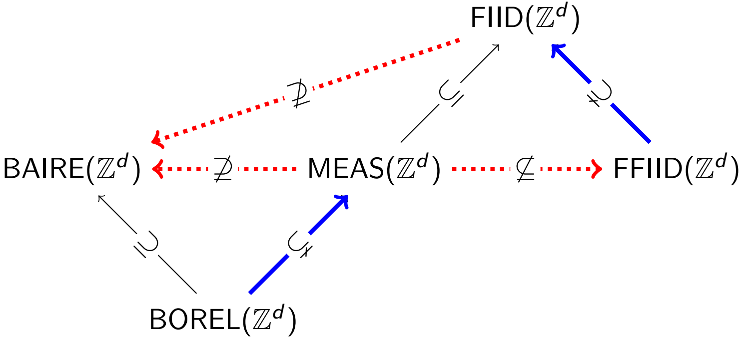

Figure: Complexity classes of LCLs on $\mathbb{Z}^d$, $d \geq 2$.

Blue arrows are strict inclusions. $\subsetneq$

Red dotted arrows are noninclusion. $\not\subseteq$

Our Results

Theorem (B.–Bernshteyn–Lyons–Weilacher)

For $d \geq 2$, there is an LCL Π on $\mathbb{Z}^d$ so that:

- $\Pi \in \mathrm{MEAS}(\mathbb{Z}^d)$,
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This is the first example of a group Γ with $\mathsf{MEAS}(\Gamma) \not\subseteq \mathsf{BAIRE}(\Gamma)$ and the first group with $\mathsf{FFIID}(\Gamma) \neq \mathsf{FIID}(\Gamma)$.

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Toast

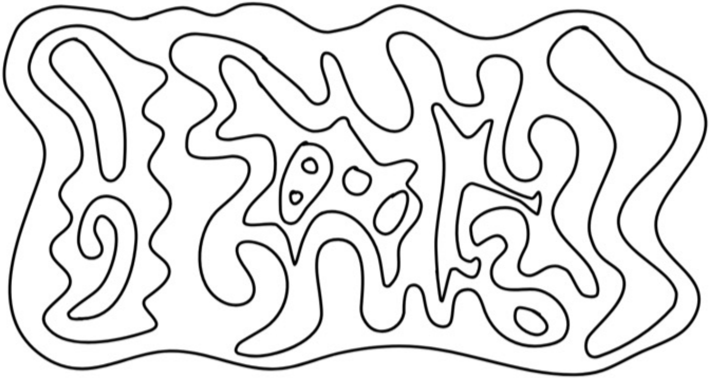

Figure 1. A few pieces of a toast.

Toast

Definition

Let $\mathbb{Z}^n \curvearrowright X$ be a free Borel action inducing a graph G. We say that a collection of finite sets $\mathcal{T} \subseteq [X]^{<\omega}$ with $\bigcup \mathcal{T} = X$ is a Borel q-**toast** if the following two conditions hold for all $K, L \in \mathcal{T}$,

- either $K \cap L = \varnothing$, $K \subseteq L$, or $L \subseteq K$,
- we have $d(\partial K, \partial L) \geq q$ in the graph metric.

Note: Borel graphs are hyperfinite iff they admit a 0-toast.

Theorem (Gao–Jackson–Krohne–Seward, 2014-2024):

Borel graphs induced by free actions of $\mathbb{Z}^d$ on a standard Borel space admit a Borel q-toast for any $q \in \mathbb{N}$.

Why toast?

Theorem (Gao–Jackson–Krohne–Seward, 2014-2024):

Borel graphs induced by free actions of $\mathbb{Z}^d$ admit a Borel proper 3-coloring.

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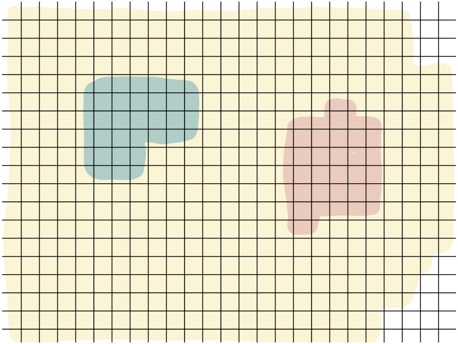

Figure: Proper coloring on $\mathbb{Z}^2$

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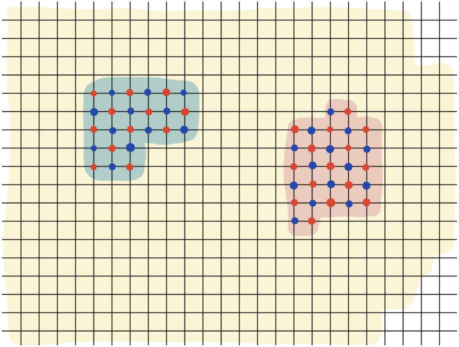

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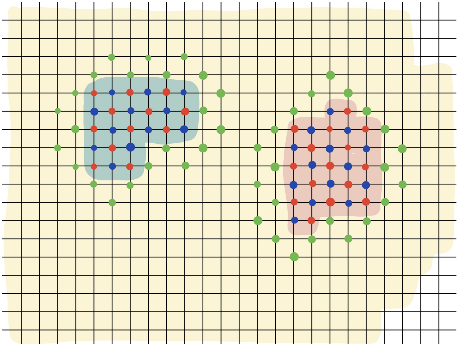

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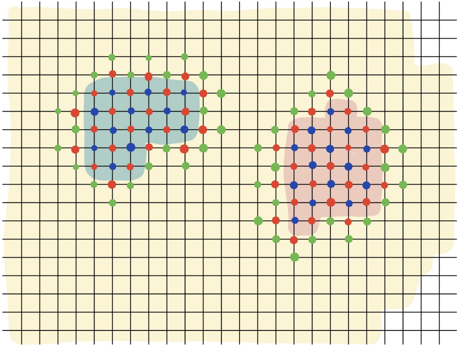

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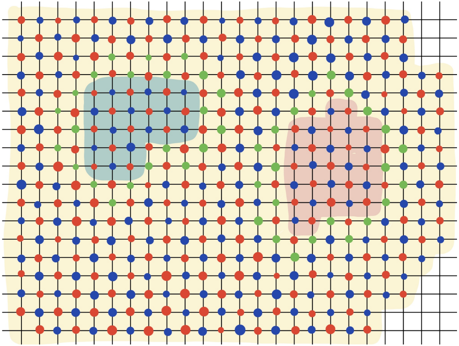

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Rectangular Toast

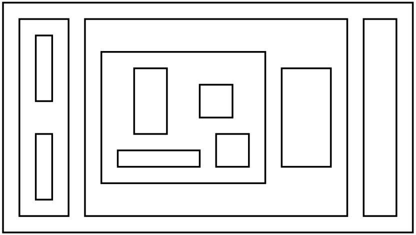

Figure 3. A rectangular toast for $\mathbb{Z}^2$.

Definition

A **rectangular** q-toast is a q-toast whose pieces are all rectangles.

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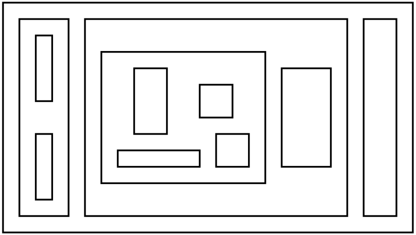

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Theorem (folklore):

Free Borel actions of $\mathbb{Z}^d$ on a standard probability space admit rectangular q-toast on a conull set (but not on a comeager set).

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What if we try to encode rectangular toast as an LCL?

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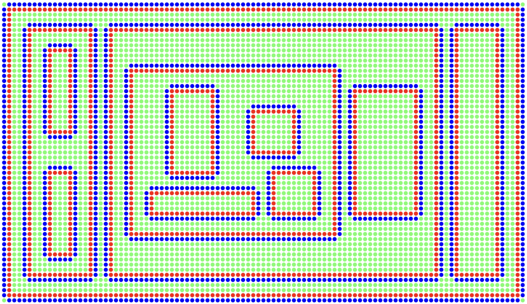

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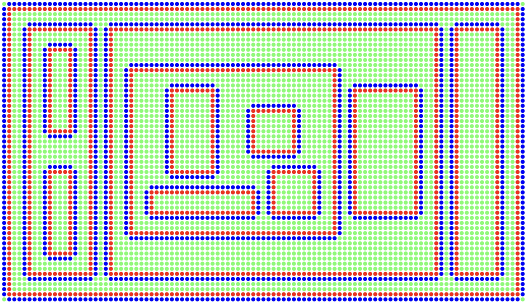
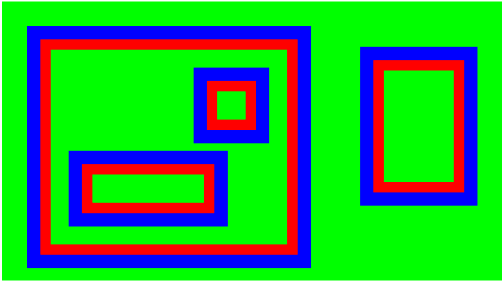
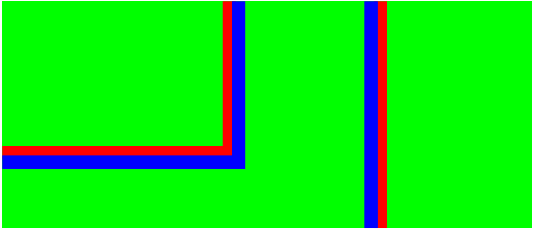


Consider the LCL whose solutions 3-color $\mathbb{Z}^d$ so it *locally* looks like the picture above.

Issues with this

These diagrams locally look the same.





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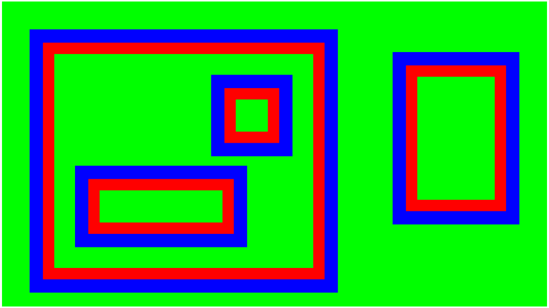
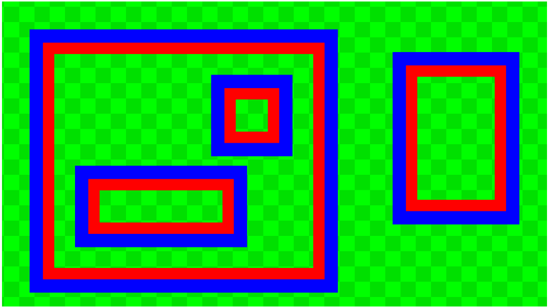


The Fix

Require the green regions to be 2-colored. This is our new LCL CRT.



We then have CRT $\in$ MEAS($\mathbb{Z}^d$) by the existence of a rectangular toast.

CRT has no Baire Measurable Solution

Theorem (B.–Bernshteyn–Weilacher–Lyons):

CRT does not always admit a Baire measurable solution.

Proof.

- Let $Z^d \curvearrowright X$ be an appropriate action. Assume for contradiction $f : X \to \{\text{RED}, \text{BLUE}, \text{GREEN0}, \text{GREEN1}\}$ be a Baire measurable solution to CRT. Let $\mathcal{T}$ be the corresponding (possibly partial) rectangular toast encoded by f.

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- Consider the generic orbit $\mathcal{O}$, which we show does not admit complete rectangular toast. Therefore we can show $X \setminus \bigcup \mathcal{T}$ is connected.

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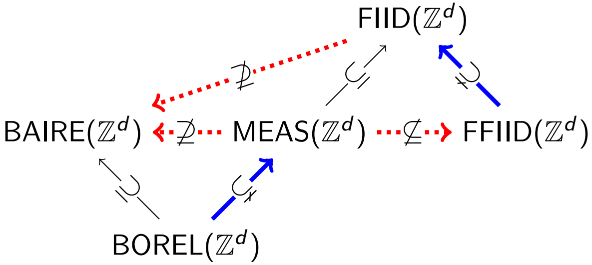
Proof.

- Let $Z^d \curvearrowright X$ be an appropriate action. Assume for contradiction $f : X \to \{\text{RED}, \text{BLUE}, \text{GREEN0}, \text{GREEN1}\}$ be a Baire measurable solution to CRT. Let $\mathcal{T}$ be the corresponding (possibly partial) rectangular toast encoded by f.
- Consider the generic orbit $\mathcal{O}$, which we show does not admit complete rectangular toast. Therefore we can show $X \setminus \bigcup \mathcal{T}$ is connected.
- Then, f can be extended uniquely to a Baire measurable 2-coloring. Contradiction.

$\square$

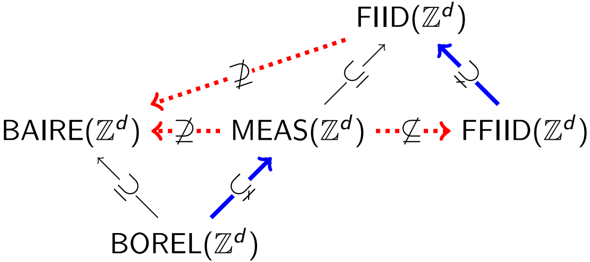
Table of Contents

Open Questions



Some arrows are still missing.

Open Questions



Some arrows are still missing. Lets make it a complete graph!

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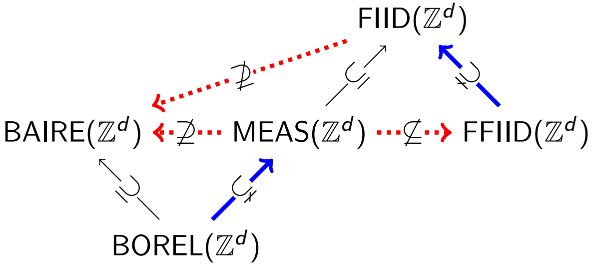

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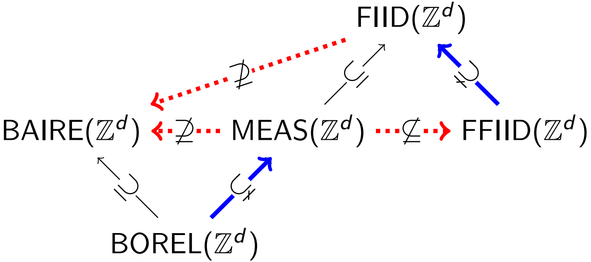

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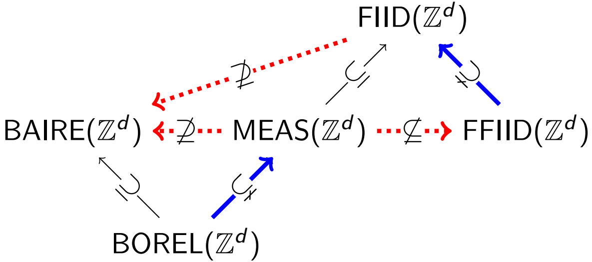

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Question: Does BAIRE($\mathbb{Z}^d$) $=$ BOREL($\mathbb{Z}^d$)?

Question: Does MEAS($\mathbb{Z}^d$) $=$ FIID($\mathbb{Z}^d$)?

Question: What is the relationship between BOREL($\mathbb{Z}^d$) and FFIID($\mathbb{Z}^d$)?

Thanks!